PHYSICS AND MATHEMATICS

# INVESTIGATION OF THE APPROXIMATION OF CONTINUOUS PERIODIC FUNCTIONS ON THE TORUS

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**ABSTRACT**

Main purpose of the present work is development of qualitative theory of difference equations in the space of bounded numeric sequences.
Main result is the establishment of necessary conditions of the existence of invariant toroidal manifolds for countable systems of differential and difference equations. In order to solve this problem, observed spaces are constructed in a special way. Necessary conditions of the existence of invariant tori for countable systems of differential and difference equations are derived.
A concept of a continuous periodic in each variable function with period $2\pi$, values of which lie in $l_2$, is introduced. Spaces, in which observations are made, are constructed in a special way. A theorem on approximation of a function from the corresponding space by trigonometric polynomials is proven.

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**Introduction.** Many problems of celestial mechanics, physics and engineering lead to the investigation of oscillations of systems described by systems of nonlinear ordinary differential equations, systems of equations in partial derivatives. The methods of studying periodic and quasiperiodic solutions of such systems are developed quite fully and are described in many fundamental writings [1, 2]. The development of technical sciences led towards an increasing interest in difference equations, which turned out to be a very convenient model for describing impulse and discrete dynamic systems, as well as systems that include digital computing devices [3]. Apart from that, difference equations arise during numerical solving of many classes of differential equations using the finite difference method.

The development of the theory of difference equations was largely due to the requirements of practical developments [4].

Wide use of numerical methods in solving differential equations, especially the finite difference method, led to the demand for a more in-depth study of difference equations.

Recently, a number of works appeared, in which new methods of qualitative analysis and construction of solutions of differential and difference equations emerging in the theory of nonlinear oscillations, are developed [5 – 7]. In connection with the new requirements of technical sciences, there is an urgent need for the construction of new methods for studying oscillatory processes and nonlinear systems.

Recently there is an increasing interest in problems related to systems of differential and difference equations in the space of bounded numeric sequences. Such systems are called counting systems. The main attention of modern studies is paid to the distribution of the above-mentioned class of systems of results that take place for finite-dimensional systems of differential and difference equations [8, 9].

**Research.** Consider functional spaces $C^r\left(\mathcal{T}_m, l_2\right)$, $H^r\left(\mathcal{T}_m, l_2\right)$.

Let $f(\varphi) = \begin{pmatrix} f^1(\varphi) \\ \dots \\ f^n(\varphi) \\ \dots \end{pmatrix}$ be a function of a variable $\varphi \in \mathcal{T}_m$, which takes values in $l_2$, continuous and periodic in each variable $\varphi_\alpha$ $(\alpha = 1,2,\dots,m)$ with period $2\pi$.

A set of such functions forms a linear space, which will be further denoted as $C\left(\mathcal{T}_m, l_2\right)$, where $\mathcal{T}_m$ is a torus, which has the dimension m. This space transforms into a complete normed space by introduction of the following norm $|f|_0 = \max\limits_{\varphi \in \mathcal{T}_m} \|f(\varphi)\|$, where $\|f\|^2 = \sum\limits_{i=1}^{\infty} \left|f_i\right|^2$ is the norm of the function $f(\varphi)$ in space $l_2$.

Denote a partial derivative of the function $f(\varphi)$ of order $\rho$ $(\rho \geq 0)$ with respect to $\varphi_\alpha$ for any $\alpha = 1,2,\dots,m$ by

$$f^{(\rho)}\left(\varphi_\alpha\right) \overset{df}{:=} \begin{pmatrix} f^{1\,(\rho)}\left(\varphi_\alpha\right) \\ \dots \\ f^{n\,(\rho)}\left(\varphi_\alpha\right) \\ \dots \end{pmatrix}$$

Consider $f^{(\rho)}\left(\varphi_\alpha\right) \in l_2$.

In $C\left(\mathcal{T}_m, l_2\right)$ select a subspace $C^r\left(\mathcal{T}_m, l_2\right)$ of those functions, which have their partial derivatives with respect to all $\varphi_\alpha$ $(\alpha = 1,2,\dots,m)$ of order less or equal than r. The set $C^r\left(\mathcal{T}_m, l_2\right)$ transforms into the complete normed space by introduction of the following norm

$$\left|f(\alpha)\right|_r = \max_{0 \leq \rho \leq r} \left| f^{(\rho)}(\varphi) \right|_0$$

where $f^{(\rho)}$ is any partial derivative with respect $\varphi_\alpha$ $(\alpha = 1,2,\dots,m)$ of order $\rho$.

Let $\mathrm{P}(\varphi)$ be a trigonometric polynomial in $l_2$, where $\varphi \in \mathcal{T}_m$, which means that

$$\mathrm{P}(\varphi) = \begin{pmatrix} \mathrm{P}^1(\varphi) \\ \dots \\ \mathrm{P}^n(\varphi) \\ \dots \end{pmatrix} \in l_2, \ \varphi \in \mathcal{T}_m,$$

where $\mathrm{P}^n(\varphi)$ is a trigonometric polynomial in $C\left(\mathcal{T}_m\right)$ - the space, studied in [10], for $\forall n \geq 1$, thus the resulting sum is

$$P^n(\varphi) = \sum_{\|k\| \le N} P_k^n e^{i(k,\varphi)} ,$$

where $k = (k_1, k_2, \dots, k_m)$ lies in the space $\mathbf{Z}^m$, elements of which are integers, $(k,\varphi) = k_1\varphi_1 + k_2\varphi_2 + \dots + k_m\varphi_m$, $P_k^n$ is a complex number, N any nonnegative integer.

The set of all polynomials with these properties forms a linear subspace, denoted by $\mathcal{P}(\mathcal{T}_m, l_2)$.

**Results of the Research.** The following theorem holds.

**Theorem 1.1.** Function $f(\varphi) \in C(\mathcal{T}_m, l_2)$ can be uniformly approximated by trigonometric polynomials, thus for $\forall$ $f(\varphi) \in C(\mathcal{T}_m, l_2)$ there exists a sequence of trigonometric polynomials $P_\nu$, $\nu = 1,2,\dots$, such that the following equality is fulfilled

$$\lim_{\nu \to \infty} \left| f(\varphi) - P_\nu(\varphi) \right|_0 = 0$$

for $\forall \varphi \in \mathcal{T}_m$.

*Proof:* Show that for $\forall f(\varphi) \in C(\mathcal{T}_m, l_2)$ and $\forall \varepsilon > 0$.

$\exists P(\varphi) \in \mathcal{P}(\mathcal{T}_m, l_2)$ such that the following relation holds

$$\left| f(\varphi) - P(\varphi) \right|_0 < \varepsilon \quad . \tag{1}$$

Construct such polynomial $P(\varphi)$.

Since $f(\varphi) \in C(\mathcal{T}_m, l_2)$, this implies $\left| f(\varphi) \right|_0 < \infty$, so there exists a number N such that

$$\sum_{n=N+1}^{\infty} \left| f^n(\varphi) \right|^2 < \frac{\varepsilon}{2} \text{, for } \forall \varphi \in \mathcal{T}_m .$$

Denote $f_N(\varphi) = \begin{pmatrix} f^1(\varphi) \\ \dots \\ f^N(\varphi) \end{pmatrix}$.

Clearly $f_N(\varphi) \in C(\mathcal{T}_m)$, where $C(\mathcal{T}_m)$ - is the space, studied in [10], as was indicated above.

By the Weierstrass theorem [1] there exists a trigonometric polynomial $P_N(\varphi) \in C(\mathcal{T}_m)$ such that

$$\max_{\varphi \in \mathcal{T}_m} \sum_{i=1}^{N} \left| f^i(\varphi) - P_N^i(\varphi) \right|^2 < \frac{\varepsilon}{2} ,$$

thus $P_N(\varphi)$ uniformly approximates $f_N(\varphi)$.

Consider in the space $\mathcal{P}\left(\mathbf{7}_m, l_2\right)$ the following polynomial

$$\mathrm{P}(\varphi) = \begin{pmatrix} \mathrm{P}_N^1(\varphi) \\ \ldots \\ \mathrm{P}_N^N(\varphi) \\ 0 \\ \ldots \end{pmatrix},$$

that is, first N coordinates of $\mathrm{P}(\varphi)$ correspond to the coordinates of $\mathrm{P}_N(\varphi)$ and all other coordinates are equal to 0. Clearly $\mathrm{P}(\varphi) \in \mathcal{P}\left(\mathbf{7}_m\right)$.

Observe this norm:

$$\left|f(\varphi) - \mathrm{P}(\varphi)\right|_0^2 = \max_{\varphi \in \mathbf{7}_m} \left\|f(\varphi) - \mathrm{P}(\varphi)\right\|^2 =$$

$$\leq \max_{\varphi \in \mathbf{7}_m} \sum_{i=1}^{\infty} \left|f^i(\varphi) - \mathrm{P}^i(\varphi)\right|^2 \leq$$

$$\leq \max_{\varphi \in \mathbf{7}_m} \left( \sum_{i=1}^{N} \left|f^i(\varphi) - \mathrm{P}^i(\varphi)\right|^2 + \sum_{i=N+1}^{\infty} \left|f^i(\varphi)\right|^2 \right) \leq$$

$$\leq \max_{\varphi \in \mathbf{7}_m} \sum_{i=1}^{N} \left|f^i(\varphi) - \mathrm{P}^i(\varphi)\right|^2 + \max_{\varphi \in \mathbf{7}_m} \sum_{i=N+1}^{\infty} \left|f^i(\varphi)\right|^2 \leq$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

By setting $\varepsilon = \frac{1}{\nu}$ for $\nu = 1,2,\ldots$, fing for each $\varepsilon = \frac{1}{\nu}$ from previous considerations a trigonometric polynomial $\mathrm{P}_\nu(\varphi)$ which satisfies (1.1).

Thus we have constructed the sequence of trigonometric polynomials

$$\mathrm{P}_\nu(\varphi), \quad \nu = 1,2,\ldots,$$

which uniformly approximates an arbitrary function $f(\varphi) \in C\left(\mathbf{7}_m, l_2\right)$ and the following equality holds

$$\lim_{\nu \to \infty} \left|f(\varphi) - \mathrm{P}_\nu(\varphi)\right|_0 = 0, \ \forall \varphi \in \mathbf{7}_m .$$

The proof is complete.

The theorem above implies that the space $C\left(\mathbf{7}_m, l_2\right)$ is the closure in norm $|\ldots|_0$ of the space of trigonometric polynomials $\mathcal{P}\left(\mathbf{7}_m, l_2\right)$.

Similar result takes place for the spaces $C^r\left(\mathbf{7}_m, l_2\right)$, where each of them is the closure in norm $|\ldots|_r$ of the space of trigonometric polynomials $\mathcal{P}\left(\mathbf{7}_m, l_2\right)$.

In that way it is possible to create a chain of Banach spaces, embedded in one another, which means that

$$C(\mathcal{T}_m,l_2)=C^0(\mathcal{T}_m,l_2)\supset C^1(\mathcal{T}_m,l_2)\supset\ldots\supset$$
$$\supset C^r(\mathcal{T}_m,l_2)\supset\ldots\supset C^\infty(\mathcal{T}_m,l_2) \qquad (2)$$

where $C^\infty(\mathcal{T}_m,l_2)=\bigcap_{r=0}^{\infty} C^r(\mathcal{T}_m,l_2)$ .

Further, for any two trigonometric polynomials in the space $\mathcal{P}(\mathcal{T}_m,l_2)$ of form

$$P=\sum_{\|k\|\le N} P_k\, e^{i(k,\varphi)} ,$$

$$Q=\sum_{\|k\|\le N} Q_k\, e^{i(k,\varphi)}$$

the scalar product $(\cdot,\cdot)_0$ can be defined by setting

$$(P,Q)_0=\frac{1}{(2\pi)^m}\int_{\mathcal{T}_m}\langle P,Q\rangle\, d\varphi_1\ldots d\varphi_m=$$

$$=\sum_k \langle P_k,Q_{-k}\rangle , \qquad (3)$$

where $\langle P_k,Q_{-k}\rangle=\sum_{j=1}^{\infty} P_k^j\, Q_{-k}^j$ is the ordinary scalar product of elements from $l_2$

$$P_k=\left(P_k^1,P_k^2,\ldots,P_k^n,\ldots\right) ,$$
$$Q_{-k}=\left(Q_{-k}^1,Q_{-k}^2,\ldots,Q_{-k}^n,\ldots\right) .$$

The product (3) in $\mathcal{P}(\mathcal{T}_m,l_2)$ induces the norm $\|\ldots\|_0$, defined as follows

$$\|P\|_0^2=(P,P)_0=\frac{1}{(2\pi)^m}\int_{\mathcal{T}_m}\|P\|^2\, d\varphi_1\ldots d\varphi_m=$$

$$=\sum_{\|k\|\le N}\left\|P_k\right\|^2 .$$

By closing $\mathcal{P}(\mathcal{T}_m,l_2)$ in that norm the Hilbert space is obtained, which is denoted by $H^0(\mathcal{T}_m,l_2)$ . Elements of that space are rows $\sum_{k\in\mathbf{Z}^m} f_k\, e^{i(k,\varphi)}$ , where the sum $\sum_{k\in\mathbf{Z}^m}\left\|f_k\right\|^2$ is finite.

For polynomials $P$ and $Q$ for any nonnegative integer r the scalar product $(\cdot,\cdot)_r$ can be defined by setting similarly to [11] the following

$$(P,Q)_r=\left((1-\Delta)^r P,Q\right)_0=$$

$$= \sum_{\|k\| \le N} \left(1 + \|k\|^2\right)^r \left\langle P_k, Q_{-k} \right\rangle , \qquad (4)$$

where $\Delta = \sum_{\nu=1}^{m} \frac{\partial^2}{\partial \varphi_\nu^2}$ is Laplace operator.

The product (3), (4) induces in the space $\mathcal{P}\left(\mathcal{T}_m, l_2\right)$ norm $\|..\|_r$, defined as follows:

$$\|P\|_r^2 = \left(P, P\right)_r = \sum_{\|k\| \le N} \left(1 + \|k\|^2\right)^r \left\|P_k\right\|^2 .$$

By closing the space $\mathcal{P}\left(\mathcal{T}_m, l_2\right)$ in the norm above the Hilbert space $H^r\left(\mathcal{T}_m, l_2\right)$ is obtained.

The elements of this space are rows $\sum_{k \in \mathbf{Z}^\mathbf{m}} f_k\, e^{i(k,\varphi)}$, for which the sum $\sum_{k \in \mathbf{Z}^\mathbf{m}} \left(1 + \|k\|^2\right)^r \left\|f_k\right\|^2$ is finite.

Starting from $\mathcal{P}\left(\mathcal{T}_m, l_2\right)$ we can construct the chain of Hilbert spaces

$$H\left(\mathcal{T}_m, l_2\right) = H^0\left(\mathcal{T}_m, l_2\right) \supset H^1\left(\mathcal{T}_m, l_2\right) \supset \ldots \supset$$

$$\supset H^r\left(\mathcal{T}_m, l_2\right) \supset \ldots \supset H^\infty\left(\mathcal{T}_m, l_2\right) , \qquad (5)$$

де $H^\infty\left(\mathcal{T}_m, l_2\right) = \bigcap_{r=0}^{\infty} H^r\left(\mathcal{T}_m, l_2\right)$ .

Similar to the result in [10] we can show that the space $H^r\left(\mathcal{T}_m, l_2\right)$ is identified with the Sobolev space $\mathcal{L}_2\left(\mathcal{T}_m, l_2\right)$ of periodic in $\varphi_\alpha$ $\left(\alpha = 1, \ldots, m\right)$ with period $2\pi$ functions, which have generalized derivatives with respect to $\varphi$ of order r and less than r.

**Conclusions.** In the present work the concept of continuous periodic for each variable with period $2\pi$ function, which has values in $l_2$, is introduced. The space of these functions with norm $|..|_0$ is denoted as $C\left(\mathcal{T}_m, l_2\right)$. Theorem 1.1 on the approximation of a function of the space $C\left(\mathcal{T}_m, l_2\right)$ by trigonometric polynomials (1), which lie in the space $\mathcal{P}\left(\mathcal{T}_m, l_2\right)$, constructed in a special way, is proven. The chain of Banach spaces (2), embedded in one another with corresponding norms is constructed:

$$C\left(\mathcal{T}_m, l_2\right) = C^0\left(\mathcal{T}_m, l_2\right) \supset C^1\left(\mathcal{T}_m, l_2\right) \supset \ldots \supset$$

$$\supset C^r\left(\mathcal{T}_m, l_2\right) \supset \ldots \supset C^\infty\left(\mathcal{T}_m, l_2\right) .$$

Additionally the norm $\|..\|_0$ is introduced. Closure of the space $\mathcal{P}(\mathcal{T}_m, l_2)$ in norm $\|..\|_0$ is Hilbert space $H(\mathcal{T}_m, l_2)$. Considering analogously, the chain of Hilbert spaces (5), embedded in one another with corresponding norms is obtained:

$$H(\mathcal{T}_m, l_2) = H^0(\mathcal{T}_m, l_2) \supset H^1(\mathcal{T}_m, l_2) \supset \ldots \supset$$

$$\supset H^r(\mathcal{T}_m, l_2) \supset \ldots \supset H^\infty(\mathcal{T}_m, l_2) \ .$$

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