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INTERVAL EDGE-COLORING OF COMPLETE AND COMPLETE BIPARTITE GRAPHS WITH RESTRICTIONS

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ABSTRACT

An edge-coloring of a graph *G* with consecutive integers $c_1, \ldots, c_t$ is called an interval t-coloring, if all colors are used, and the colors of edges incident to any vertex of *G* are distinct and form an interval of integers. A graph *G* is interval colorable if it has an interval t-coloring for some positive integer *t*. In this paper, we consider the case where there are restrictions on the edges, and the edge-coloring should satisfy these restrictions. We show that the problem is NP-complete for complete and complete bipartite graphs. We also provide a polynomial solution for a subclass of complete bipartite graphs when the restrictions are on the vertices.

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Introduction. All graphs considered in this paper are undirected, finite, and have no loops or multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$. The maximum degree of vertices in G is denoted by $\Delta(G)$.

A complete graph [1] is a graph in which every pair of distinct vertices is connected by an edge. The complete graph having n vertices is denoted by K_n. A bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets U_1 and U_2 such that every edge connects a vertex in U_1 to a vertex in U_2. A complete bipartite graph is a bipartite graph such that two vertices are adjacent, if and only if they are in different partite sets. When the sets have sizes n and m, the complete bipartite graph is denoted by $K_{n,m}$.

An edge-coloring of a graph G is an assignment of colors to the edges of the graph so that no two adjacent edges have the same color. An edge-coloring of a graph G with colors $1, \ldots, t$ is an interval t-coloring if all colors are used, and the colors of edges incident to each vertex of G form an interval of integers. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. The concept of an interval edge-coloring of a graph was introduced by Asratian and Kamalian [2]. This means that an interval t-coloring is a function $\alpha: E(G) \to \{1, \ldots, t\}$ such that for each edge e the color $\alpha(e)$ of that edge is an integer from 1 to t, for each color from 1 to t there is an edge with that color and for each vertex v all the edges incident to v have different colors forming an interval of integers. For a graph $G \in \mathfrak{N}$, the least and the greatest values of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively.

The set of integers $\{a, a+1, \ldots, b\}$, $a \leq b$, is denoted by $[a, b]$. Let I_k be the set $[1, k]$ of integers, then 2^{I_k} is the set of all the subsets of I_k. We will denote by $\tau(I_k)$ the set of all the elements from 2^{I_k} that are an interval of integers. More formally $\tau(I_k) = \{s: s \in 2^{I_k}, s\ is\ a\ non\ empty\ interval\ of\ integers\}$. The greatest common divisor of two positive integers a, b is denoted by $\sigma(a, b)$.

For an interval coloring, α and a vertex v, the set of all the colors of the incident edges of v is called the spectrum of that vertex in α and is denoted by $S_\alpha(v)$. The smallest and the largest numbers in $S_\alpha(v)$ are denoted by $\underline{S_\alpha}(v)$ and $\overline{S_\alpha}(v)$, respectively.

We consider the following two problems:

Problem 1: Given a graph G and for some k restrictions on the edges $R: E(G) \to 2^{I_k}$. Find an interval edge-coloring $\alpha: E(G) \to I_k$ such that $\alpha(e) \in R(e)$ for all $e \in E(G)$.

Problem 2: Given a graph G and for some k restrictions on the vertices $L: V(G) \to 2^{I_k}$. Find an interval edge-coloring $\alpha: E(G) \to I_k$ such that $S_\alpha(v) \subseteq L(v)$ for all $v \in V(G)$.

We show that the Problem 1 is NP-complete for complete and complete bipartite graphs. For complete bipartite graphs we provide a polynomial solution for the Problem 2 when $\sigma(n, m) = 1$.

Interval edge-colorings have been intensively studied in different papers. Lower and upper bounds on the number of colors in interval edge-colorings were provided in [3, 4] and the bounds were improved for different graphs: planar graphs [5], r-regular graphs with at least $2 \cdot r + 2$ vertices [6], cycles, trees, complete bipartite graphs [3], n-dimensional cubes [7,8], complete graphs [9, 10], Harary graphs [11], complete k-partite graphs [12], even block graphs [13]. In [14], interval edge-colorings with restrictions on edges were considered. In this case there can be restrictions on the edges for the allowed colors. In [15, 16], interval edge-colorings with restrictions on the spectrums were considered.

NP-completeness of interval edge-coloring with restrictions on edges for complete bipartite graphs

Here we consider the Problem 1 for complete bipartite graphs $K_{n,m}$. We will show that the problem is NP-complete even for the case of $K_{n,n}$ where the restrictions are from $[1, n]$. Finding an interval n-coloring that meets the restrictions, is the same as finding an edge-coloring that meets the restrictions, since $d_{K_{n,n}}(v) = n$ for all $v \in V(K_{n,n})$ and all the spectrums are going to be the interval $[1, n]$. The problem becomes the following:

Problem 3: Given a complete bipartite graph $K_{n,n}$ and some restrictions on the edges $R: E(K_{n,n}) \to 2^{I_n}$. Find an edge-coloring $\alpha: E(K_{n,n}) \to I_n$ such that $\alpha(e) \in R(e)$ for all $e \in E(K_{n,n})$.

A Latin square is an $n \times n$ matrix M with entries from the set $\{1, \ldots, n\}$ such that no column or row contains any repeated entry.

A partial Latin square is an $n \times n$ matrix M with entries from the set $\{0, 1, \ldots, n\}$ such that no column or row contains any repeated entry other than 0.

Problem 4: Let M be an $n \times n$ partial Latin square. Is it possible to extend M to a Latin square, i.e., can we replace each zero entry in M by an element of $\{1,2, \ldots, n\}$ in such a way that no row or column contains a repeated entry?

In [17], Colbourn showed that the Problem 4 is NP-complete. We now show that the Problem 3 is NP-complete too.

Theorem 1: The Problem 3 of finding an edge-coloring for a $K_{n,n}$ with given restrictions $R: E(K_{n,n}) \to 2^{I_n}$ is NP-complete.

Let M be an $n \times n$ partial Latin square. Let $M_{r,c}$ be the element in the row r and the column c. We will create a complete bipartite graph $G = K_{n,n}$ with restrictions $R: E(G) \to 2^{I_n}$ such that finding an edge-coloring in that graph is equivalent to extending the partial Latin square into a Latin square. Let $V_1 = \{u_1, \ldots, u_n\}$ be the set of vertices in the first part ($|V_1| = n$), and let $V_2 = \{v_1, \ldots, v_n\}$ be the set of vertices in the second part ($|V_2| = n$). The vertices V_1 will represent the rows of the matrix M and the vertices V_2 will represent the columns of the matrix M. Let $c = M_{i,j}$. If $c \neq 0$ then we take $R(u_i, v_j) = \{c\}$ (we should use the color c for the edge (u_i, v_j)). If $c = 0$ then we can take $R(u_i, v_j) = [1, n]$. Note that we could remove from $R(u_i, v_j)$ all the other colors that appeared in that row or that column, but since we are going to find an edge-coloring we do not have to do it.

If $\bar{M}$ is an extended Latin square of M, then taking $\alpha(v_i, v_j) = \bar{M}_{i,j}$ satisfies the restrictions R and is an edge-coloring (since in a Latin square the elements of each row and each column are different).

Let α be an edge-coloring that satisfies the restrictions R. Taking $\bar{M}_{i,j} = \alpha(u_i, v_j)$ satisfies the restrictions of Latin square since the colors are from $[1, n]$, the existing colors of M are colored in their respective colors, and for each row and each column, the elements are different (since they are the incident edges of a vertex from $K_{n,m}$). Hence the Problem 3 is NP-complete.

Corollary 1: The Problem 1 of finding an interval edge-coloring for a complete bipartite graph $K_{n,m}$ with given restrictions $R: E(K_{n,m}) \to 2^{I_{n+m-1}}$ is NP-complete.

The Problem 3 is a special case of finding an interval edge-coloring for complete bipartite graphs. From the Theorem 1 we know that the Problem 3 is NP-complete, hence the more general problem is also NP-complete.

NP-completeness of interval edge-coloring with restrictions on edges for complete graphs

Here we consider the Problem 1 for complete graphs and show that the general problem is NP-complete. We will show that the problem is NP-complete even for the case of K_n where the restrictions are from $[1, n-1]$. Finding an interval $(n-1)$-coloring that meets the restrictions, is the same as finding an edge-coloring that meets the restrictions, since $d_{K_n}(v) = n-1$ for all $v \in V(K_n)$ and all the spectrums are going to be the interval $[1, n-1]$. The problem becomes the following:

Problem 5: Given a complete graph K_n and some restrictions on the edges $R: E(K_n) \to 2^{I_{n-1}}$. Find an edge-coloring $\alpha: E(K_n) \to I_{n-1}$ such that $\alpha(e) \in R(e)$ for all $e \in E(K_n)$.

From [4], it is known that $w(K_{2\cdot m}) = 2 \cdot m - 1 = \Delta(K_{2\cdot m})$ and $K_{2\cdot m+1}$ is not interval colorable. Hence we are interested only in $n = 2 \cdot m$. Any edge-coloring $\alpha: E(K_{2\cdot m}) \to I_{2\cdot m-1}$ can be represented with a matrix M, such that $M_{i,j} = \alpha(v_i, v_j)$ $(v_i, v_j \in V(K_{2\cdot m}))$, and $M_{i,i} = 2 \cdot m$ (we color the diagonal with the color $2 \cdot m$ to have a full matrix). Note that M is a symmetric Latin square. Any symmetric $2 \cdot m \times 2 \cdot m$ Latin square M with diagonal elements equal to $2 \cdot m$ can be transformed into an edge-coloring of $K_{2\cdot m}$ by taking $\alpha(v_i, v_j) = M_{i,j}$.

Definition: Let $n = 2 \cdot m$. An n-diagonal Latin square is a symmetric $n \times n$ Latin square M such that $M_{i,i} = n$ for all $1 \le i \le n$.

Note that n-diagonal Latin squares are only defined for even n. In fact, any symmetric Latin square with odd n should contain all the colors $1, \dots, n$ in its diagonal, since each number $1, \dots, n$ should be used an odd number of times, and in a symmetric matrix, their counts are the same above and below the diagonal.

The problem of completing an n-diagonal partial Latin square can be reduced to the Problem 5. For an n-diagonal partial Latin square M we can construct a complete graph K_n and restrictions R the following way: Let $c = M_{i,j}$ $(i \ne j)$. If $c \ne 0$ then we take $R(v_i, v_j) = \{c\}$, otherwise, if $c = 0$ then we take $R(v_i, v_j) = [1, n-1]$. In this case finding an edge-coloring α that meets the restrictions is equivalent to completing M. To construct the Latin square $\bar{M}$ from α we do the following: we take $\bar{M}_{i,i} = n$, and we take $\bar{M}_{i,j} = \alpha(v_i, v_j)$ if $i \ne j$. Since α is an edge-coloring $\bar{M}$ is a Latin square. Similarly, if we have the completed Latin square $\bar{M}$ we can construct the edge-coloring α by taking $\alpha(v_i, v_j) = \bar{M}_{i,j}$.

In [18], it was shown that the problem of completing a symmetric partial Latin square is NP-complete, but since the n-diagonal Latin squares are a subclass of symmetric Latin squares, we will provide proof for this case.

Theorem 2: The problem of completing an n-diagonal partial Latin square is NP-complete.

We will reduce the problem of completing a symmetric partial Latin square of order $2 \cdot m - 1$ to the problem of completing a $2 \cdot m$-diagonal partial Latin square.

Let M be any $(2 \cdot m - 1) \times (2 \cdot m - 1)$ symmetric partial Latin square. We construct a new $(2 \cdot m) \times (2 \cdot m)$ matrix T by taking the diagonal of M and adding it as the last row and the last column of T and by assigning the value $2 \cdot m$ to the elements in the diagonal of T. Fig. 1 illustrates that transformation.

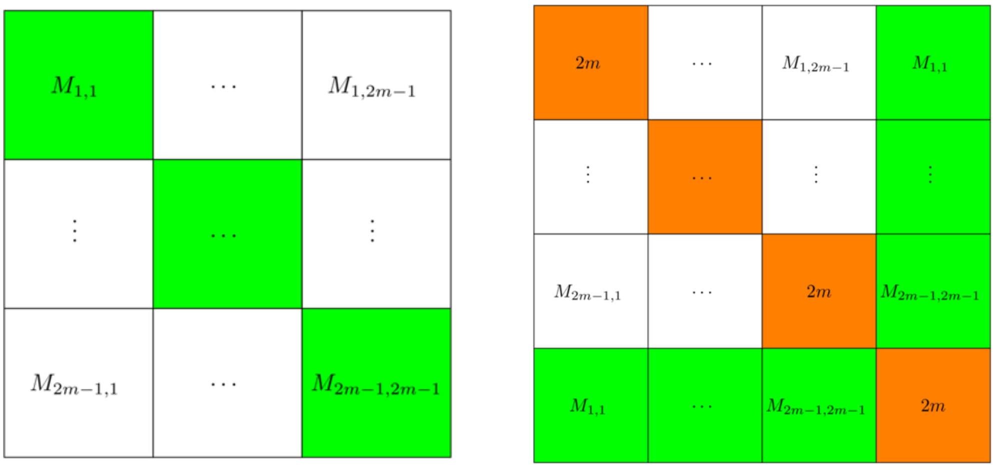

Fig. 1. The transformation of a symmetric partial Latin square M to a $2 \cdot m$-diagonal partial Latin square T.

Since T should be symmetric, the last row will be the same as the last column, and we can later transform completed T to a completed M by taking the diagonal of M the last row of T (without the last element). Hence completing the $2 \cdot m$-diagonal partial Latin square T is equivalent to completing the symmetric partial Latin square M, which means the problem of completing an n-diagonal partial Latin square is NP-complete.

Corollary 2: The Problem 5 of finding an edge-coloring of a complete graph K_n with given restrictions $R: E(K_n) \to 2^{I_{n-1}}$ is NP-complete.

This follows from the fact that the problem of completing an n-diagonal prtial Latin square can be reduced to the Problem 5. From the Theorem 2 the problem of completing an n-diagonal partial Latin square is NP-complete, hence the Problem 5 is also NP-complete.

Corollary 3: Let $t = E(K_{2\cdot m})$. The Problem 1 of finding an interval edge-coloring of a complete graph $K_{2\cdot m}$ with given restrictions $R: E(K_{2\cdot m}) \to 2^{I_t}$ is NP-complete.

The Problem 5 is a special case of finding an interval edge-coloring of complete graphs with given restrictions R on the edges. From the Corollary 2, the Problem 5 is NP-complete, which means the general problem is also NP-complete.

Interval edge-coloring of complete bipartite graphs with restrictions on vertices

Here we consider the Problem 2 for complete bipartite graphs $K_{n,m}$ where $\sigma(n,m) = 1$. From the Theorem 1 of [3], for $K_{n,m}$ it is known that $w(K_{n,m}) = n + m - \sigma(n,m)$ and $W(K_{n,m}) = n + m - 1$. This means that if $\sigma(n,m) = 1$ then the number of colors should be $t = n + m - 1$. For simplicity, we will assume that the restrictions L on the vertices are from $[1,t]$ ($L(v) \subseteq [1,t]$ for all $v \in V(K_{n,m})$). The problem becomes the following:

Problem 6: Given a complete bipartite graph $G = K_{n,m}$ with $\sigma(n,m) = 1$ and given restrictions on the vertices $L: V(G) \to 2^{I_t}$ $(t = n + m - 1)$. Find an interval edge-coloring $\alpha: E(G) \to I_t$ such that $S_\alpha(v) \subseteq L(v)$ for all $v \in V(G)$.

Let V_1 be the set of vertices in the first part ($|V_1| = n$), and let $|V_2|$ be the set of vertices in the second part ($|V_2| = m$). Let α be any interval edge-coloring of $K_{n,m}$ with $n + m - 1$ colors.

If we take the vertices of V_1 and sort them in the ascending order of $\underline{S_\alpha}(u)$ then the spectrums will look like this: $[1,m], [2,m+1], \dots, [n, n+m-1]$. Similarly, if we take the vertices of V_2 and sort them in the ascending order of $\underline{S_\alpha}(v)$ then the spectrums will look like this: $[1,n], [2,n+1], \dots, [m, n+m-1]$. It means that for each part, the spectrums differ from each other. Now imagine that for the vertices $u_1, \dots, u_n$ $(u_i \in V_1)$ we know the spectrums are $[1,m], \dots, [n, n+m-1]$ and for the vertices $v_1, \dots, v_m$ $(v_i \in V_2)$ we know the spectrums are $[1,n], [2,n+1], \dots, [m, n+m-1]$ then we can construct an interval edge-coloring α such that all the spectrums are correct. The coloring could be $\alpha(u_i, v_j) = i + j - 1$. It is easy to check that this coloring is an interval edge-coloring and satisfies the spectrum restrictions.

We can solve the problem with restrictions L independently for the two parts. In the first part, we need to find vertices $u_{p_1}, \dots, u_{p_n}$such that $[i, i+m-1] \subseteq L(u_{p_i})$. In the second part we need to find vertices $v_{q_1}, \dots, v_{q_m}$ such that $[i, i+n-1] \subseteq L(v_{q_i})$. Since these are similar problems, we will show how to solve the problem for one part. The problem becomes the following:

Problem 7: Given vertices $u_1, \dots, u_n$ and for each of the vertices a list of colors $L(u_i) \subseteq [1, n+m-1]$. Find a permutation $p_1, \dots, p_n$ of the indices $1, \dots, n$ such that $[i, i+m-1] \subseteq L(u_{p_i})$ for all $1 \le i \le n$ or determine that there is no such permutation.

We will construct a bipartite graph F the following way: The left part of the graph will be the vertices $u_1, \dots, u_n$, the right part will be the vertices $g_1, \dots, g_n$ (where g_i represents the interval $[i, i+m-1]$). For each $1 \le i \le n$ and $1 \le j \le n$ we will connect the vertex u_i with the vertex g_j if and only if $[j, j+m-1] \subseteq L(u_i)$. Fig. 2 illustrates the bipartite graph F.

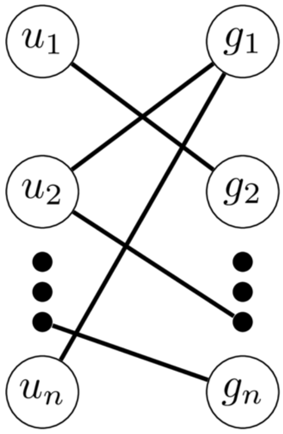

Fig. 2. The bipartite graph F constructed by the vertices $u_1, \dots, u_n$ and the intervals $g_1 = [1, m], \dots, g_n = [n, n+m-1]$.

If we can find a perfect matching [19] in F, then it is possible to find indices $p_1, \dots, p_n$ such that $[i, i+m-1] \subseteq L(u_{p_i})$ for all $1 \le i \le n$. The edges (u_{p_i}, g_i), $1 \le i \le n$ will be all the edges of the matching.

The complexity of the solution for the Problem 7 will be $O(n^3)$ for finding a perfect matching. Note that for each list $L(u_i)$ we can find in $O(n)$ all the intervals $[j, j+m-1] \subseteq L(u_i)$ using a sweep line algorithm [20]. For the Problem 7 the complexity of the algorithm will be $O(n^3 + m^3)$.

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